Exercise 42

Show that the plane that passes through the three points $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, and $C = (c_1, c_2, c_3)$ consists of the points P $= (x, y, z)$ given by

$$\begin{vmatrix} a_1 - x & a_2 - y & a_3 - z \\ b_1 - x & b_2 - y & b_3 - z \\ c_1 - x & c_2 - y & c_3 - z \end{vmatrix} = 0.$$

(Hint: Write the determinant as a triple product.)

Solution

The equation of a plane is given by

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0,$$

where $\mathbf{n}$ is a normal vector, $\mathbf{r} = (x, y, z)$, and $\mathbf{r}_0$ is a position vector for any point lying in the plane. Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$. The normal vector is obtained by taking the cross product of two displacement vectors, $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$, for example.

$$\mathbf{n} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$$

Since the displacements are taken relative to $\mathbf{a}$, set $\mathbf{r}_0 = \mathbf{a}$. The equation for the plane is then

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \quad \rightarrow \quad [(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})] \cdot (\mathbf{r} - \mathbf{a}) = 0.$$

Use the fact that the dot product is commutative.

$$(\mathbf{r} - \mathbf{a}) \cdot [(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})] = 0$$

Multiply both sides by -1.

$$(\mathbf{a} - \mathbf{r}) \cdot [(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})] = 0$$

Simplify the expression in square brackets.

$$(\mathbf{a} - \mathbf{r}) \cdot [\mathbf{b} \times (\mathbf{c} - \mathbf{a}) - \mathbf{a} \times (\mathbf{c} - \mathbf{a})] = 0$$

$$(\mathbf{a} - \mathbf{r}) \cdot (\mathbf{b} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} - \mathbf{a} \times \mathbf{c} + \underbrace{\mathbf{a} \times \mathbf{a}}_{=\mathbf{0}}) = 0$$

$$(\mathbf{a} - \mathbf{r}) \cdot (\mathbf{b} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} - \mathbf{a} \times \mathbf{c}) = 0$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} - \mathbf{a} \times \mathbf{c}) - \mathbf{r} \cdot (\mathbf{b} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} - \mathbf{a} \times \mathbf{c}) = 0$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) - \underbrace{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{a})}_{=\mathbf{0}} - \underbrace{\mathbf{a} \cdot (\mathbf{a} \times \mathbf{c})}_{=\mathbf{0}} - \mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) + \mathbf{r} \cdot (\mathbf{b} \times \mathbf{a}) + \mathbf{r} \cdot (\mathbf{a} \times \mathbf{c}) = 0$$

Both $\mathbf{b} \times \mathbf{a}$ and $\mathbf{a} \times \mathbf{c}$ are perpendicular to $\mathbf{a}$, so $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) = \mathbf{0}$ and $\mathbf{a} \times \mathbf{c} = \mathbf{0}$.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) - \mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) + \mathbf{r} \cdot (\mathbf{b} \times \mathbf{a}) + \mathbf{r} \cdot (\mathbf{a} \times \mathbf{c}) = 0$$

Use the result from part (a) of Exercise 24: $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})$ and $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$ for $\mathbf{r} \cdot (\mathbf{b} \times \mathbf{a})$ and $\mathbf{r} \cdot (\mathbf{a} \times \mathbf{c})$, respectively.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) - \mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) - \mathbf{a} \cdot (\mathbf{b} \times \mathbf{r}) - \mathbf{a} \cdot (\mathbf{r} \times \mathbf{c}) = 0$$

Seeing as how $\mathbf{b}\times\mathbf{r}$ and $\mathbf{r}\times\mathbf{c}$ are both perpendicular to $\mathbf{r}$, $\mathbf{r}\cdot(\mathbf{b}\times\mathbf{r}) = \mathbf{0}$ and $\mathbf{r}\cdot(\mathbf{r}\times\mathbf{c}) = \mathbf{0}$ can be added to the left side.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) - \mathbf{r}\cdot(\mathbf{b}\times\mathbf{c}) - \mathbf{a}\cdot(\mathbf{b}\times\mathbf{r}) - \mathbf{a}\cdot(\mathbf{r}\times\mathbf{c}) + \mathbf{r}\cdot(\mathbf{b}\times\mathbf{r}) + \mathbf{r}\cdot(\mathbf{r}\times\mathbf{c}) = 0$$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c} - \mathbf{b}\times\mathbf{r} - \mathbf{r}\times\mathbf{c}) - \mathbf{r}\cdot(\mathbf{b}\times\mathbf{c} - \mathbf{b}\times\mathbf{r} - \mathbf{r}\times\mathbf{c}) = 0$$

$$(\mathbf{a}-\mathbf{r})\cdot(\mathbf{b}\times\mathbf{c} - \mathbf{b}\times\mathbf{r} - \mathbf{r}\times\mathbf{c}) = 0$$

$$(\mathbf{a}-\mathbf{r})\cdot(\mathbf{b}\times\mathbf{c} - \mathbf{b}\times\mathbf{r} - \mathbf{r}\times\mathbf{c} + \underbrace{\mathbf{r}\times\mathbf{r}}_{=\mathbf{0}}) = 0$$

$$(\mathbf{a}-\mathbf{r})\cdot[\mathbf{b}\times(\mathbf{c}-\mathbf{r}) - \mathbf{r}\times(\mathbf{c}-\mathbf{r})] = 0$$

$$(\mathbf{a}-\mathbf{r})\cdot[(\mathbf{b}-\mathbf{r})\times(\mathbf{c}-\mathbf{r})] = 0$$

$$(a_1 - x, a_2 - y, a_3 - z)\cdot[(b_1 - x, b_2 - y, b_3 - z)\times(c_1 - x, c_2 - y, c_3 - z)] = 0$$

$$(a_1 - x, a_2 - y, a_3 - z)\cdot\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ b_1 - x & b_2 - y & b_3 - z \\ c_1 - x & c_2 - y & c_3 - z \end{vmatrix} = 0$$

Therefore, taking the dot product, the plane that passes through the three points $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, and $C = (c_1, c_2, c_3)$ can be expressed as

$$\begin{vmatrix} a_1 - x & a_2 - y & a_3 - z \\ b_1 - x & b_2 - y & b_3 - z \\ c_1 - x & c_2 - y & c_3 - z \end{vmatrix} = 0.$$